Toeplitz Operators

Definition Matrix of an operator w.r.t. an orthonormal basis in a Hilbert space H.

Let $A \in B(H)$ and let $\{e_n\}_{n \in I}$ be an orthonormal basis of H.

The matrix of A w.r.t. $\{e_n\}_{n\in I}$ is the matrix

$$[a_{mn}] \quad \text{where} \quad a_{mn} = \langle Ae_n, e_m \rangle, \quad n, m \in I$$

Theorem 1 Let $\phi \in L^\infty(\mathbb{T})$ with Fourier series $\sum_{n=-\infty}^{\infty} \alpha_n e^{in\theta}$

Then the matrix of the multiplication operator M_ϕ on L^2 w.r.t. the orthonormal basis $\{e^{in\theta}\}_{n=-\infty}^{\infty}$ is the doubly infinite matrix

$$\begin{bmatrix} \ddots & \ddots & \ddots & & & \\ & \alpha_1 & \alpha_0 & \alpha_{-1} & & \\ & & \alpha_1 & \alpha_0 & \alpha_{-1} & \\ & & & \alpha_1 & \alpha_0 & \alpha_{-1} \\ & & & & \alpha_1 & \ddots \\ & & & & & \ddots \end{bmatrix}$$

Proof. To determine the matrix of M_φ w.r.t. $\{e^{in\theta}\}_{n=-\infty}^{\infty}$ we compute for each pair of integers (m,n) the $(m,n)^{th}$ entry a_{mn} of the matrix of M_φ

Now

$$a_{mn} = \langle M_\varphi e^{in\theta}, e^{im\theta} \rangle_{L^2}$$

$$= \frac{1}{2\pi} \int_0^{2\pi} M_\varphi e^{in\theta} e^{-im\theta} \, d\theta$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \varphi(e^{i\theta}) \, e^{i(n-m)\theta} \, d\theta$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \varphi(e^{i\theta}) \, e^{-i(m-n)\theta} \, d\theta$$

$$= \alpha_{m-n}$$

Thus ~~whenever m~~ for each fixed integer k

$$a_{mn} = \alpha_{m-m} = \alpha_k \iff m-m = k$$

Hence on the main diagonal we have $a_{nn} = \alpha_{n-n} = \alpha_0$ and the diagonal just above the main we have

$a_{n+1,n} = \alpha_{n-(n+1)} = \alpha_{-1}$

and on the diagonal just below the main a typical entry shall be $a_{n,n+1} = \alpha_{n+1-n} = \alpha_1$

Similarly two diagonals above the main each entry shall be $a_{n+2;n} = \alpha_{n-(n+2)} = \alpha_{-2}$

and two entries below the main diagonal each entry shall be $a_{n,n+2} = \alpha_{n+2-n} = \alpha_2$

Hence across each diagonal we shall have a constant entry in the following manner

On the main diagonal $D_0 = (\cdots, \alpha_0, \cdots)$

i.e. each entry is α_0 and the $(0,0)^{th}$ position (i.e. centre of the entries) is α_0

On the diagonals above D_k, $k = 1, 2, \ldots$

each $D_k = (\cdots, \alpha_{-k}, \cdots)$ i.e each entry is α_{-k} for the diagonal k diagonals above D_0. Similarly for D_{-k} - The diagonals k ~~ro~~ diagonals below D_0, $\cancel{D_{-k} = (\cdots, \alpha_k}$

$D_k = (\cdots, \alpha_k, \cdots)$ i.e. each entry is α_k.

Definition Toeplitz Matrix.

A finite square matrix is called a Toeplitz matrix if it is constant along each diagonal

Example.

$$\begin{bmatrix} a & \alpha & \beta & \gamma & \delta \\ b & a & \alpha & \beta & \gamma \\ c & b & a & \alpha & \beta \\ d & c & b & a & \alpha \\ e & d & c & b & a \end{bmatrix}$$

i.e. $a_{m_1,n_1} = a_{m_2,n_2}$ whenever $m_1 - n_1 = m_2 - n_2$

A singly infinite matrix $[a_{m,n}]$ $m = 0,1,2,\dots$, $n = 0,1,2,\dots$ is called a Toeplitz matrix if each entry along any diagonal is constant

$$\text{i.e. } a_{m_1,n_1} = a_{m_2,n_2}$$

$$\Longleftrightarrow \quad m_1 - n_1 = m_2 - n_2$$

A doubly infinite matrix $[a_{m,n}]$, $m = \dots,-3,-2,-1,0,1,\dots$
$n = \dots,-3,-2,-1,0,1,2,\dots$

is called a Toeplitz matrix if its entries are constant along each diagonal

i.e. $$a_{m_1,n_1} = a_{m_2,n_2}$$

whenever $$m_1 - n_1 = m_2 - n_2$$

Theorem A bounded linear operator on $L^2(\mathbb{T})$ is multiplication by an L^∞ function ϕ if and only if its matrix, w.r.t. the standard orthonormal basis $\{e^{in\theta}\}_{n=-\infty}^{\infty}$ of L^2, is a doubly infinite Toeplitz matrix.

Proof. We have shown in Theorem 1 (page T1) that M_ϕ – multiplication by ϕ on L^2 – has a doubly infinite Toeplitz matrix.

Conversely Assume the matrix (doubly infinite) A

where $A = [a_{m,n}]$ $\quad m = \ldots,-3,-2,-1,0,1,2,\ldots$
$\quad n = \ldots,-3,-2,-1,0,1,2,\ldots$

is a doubly infinite Toeplitz matrix.

We shall produce a function $\phi \in L^\infty$ s.t.

$A = M_\phi$ (multiplication by ϕ on L^2)

Now, it is given that A is a bounded linear operator on L^2.

We also know that if T is any bounded operator on L^2 such that $AS = SA$ (where S is the bilateral shift on L^2 i.e. $Sf = e^{i\theta} f(e^{i\theta})$)

then $T = M_\phi$ for some $\phi \in L^\infty$.

So all that we have to do is show that $AS = SA$.

<u>i.e. we have to show</u>

$$\langle ASe^{in\theta}, e^{im\theta} \rangle = \langle SAe^{in\theta}, e^{im\theta} \rangle$$

Now,

$$\langle ASe^{in\theta}, e^{im\theta}\rangle = \langle Ae^{i(n+1)\theta}, e^{im\theta}\rangle$$

$$= \langle Ae^{in\theta}, e^{i(m-1)\theta}\rangle \quad \cdots \underline{\underline{(1)}}$$

($\because$ A has a Toeplitz matrix)

Thus $\langle Ae^{in\theta}, e^{i(m-1)\theta}\rangle = \langle Ae^{in\theta}, S^{-1}e^{im\theta}\rangle$

$$= \langle Ae^{in\theta}, S^{*}e^{im\theta}\rangle$$

($\because$ on L^2, S is unitary)

$$= \langle SAe^{in\theta}, e^{im\theta}\rangle \quad \cdots \underline{\underline{(2)}}$$

From $\underline{\underline{(1)}}$ & $\underline{\underline{(2)}}$ we get

$$\langle ASe^{in\theta}, e^{im\theta}\rangle = \langle SAe^{in\theta}, e^{im\theta}\rangle$$

for all integers n, m

Hence $AS = SA$

This means $A = M_\phi$ for some $\phi \in L^\infty$.

Definition Essential Range of a function ϕ in L^∞

Let $\phi \in L^\infty$. The essential range of ϕ is denoted by ess ran ϕ and defined as

$$\cancel{\text{ess ran } \phi = \{\lambda : m\{e^{i\theta} : |\phi(e^{i\theta}) - \lambda| < \epsilon\} > 0, \forall \epsilon > 0\}}$$

$$\text{ess ran } \phi = \{\lambda : m\{e^{i\theta} : |\phi(e^{i\theta}) - \lambda| < \epsilon\} > 0 \ \forall \epsilon > 0\}$$

m = normalised Lebesgue measure

Example

Let $\phi : \mathbb{T} \longrightarrow \mathbb{C}$ by

$$\phi(e^{i\theta}) = \begin{cases} e^{i\theta}, & \theta \neq \pi \\ 2, & \theta = \pi \end{cases}$$

Then the ess ran $\phi = \{e^{i\theta} : \theta \neq \pi\}$

Theorem If $\phi \in L^\infty$ then $\sigma(M_\phi) = \Pi(M_\phi) = \text{ess ran } \phi$

Proof: We first show

$$\text{ess ran } \phi \subset \Pi(M_\phi)$$

So let $\lambda \in \text{ess ran } \phi$.

For each $n = 1, 2, 3, \ldots$ define $E_n = \left\{ e^{i\theta} : |\phi(e^{i\theta}) - \lambda| < \frac{1}{n} \right\}$

and let χ_{E_n} be the characteristic function of E_n.

Clearly, $m(E_n) > 0$ since $\lambda \in \text{ess ran } \phi$.

Now $\|(M_\phi - \lambda I)\chi_{E_n}\|^2_{L^2}$

$$= \frac{1}{2\pi}\int_0^{2\pi} |(\phi(e^{i\theta}) - \lambda)\chi_{E_n}(e^{i\theta})|^2 \, d\theta$$

$$= \frac{1}{2\pi}\int_{E_n} |\phi(e^{i\theta}) - \lambda|^2 \, d\theta$$

$$\leq \frac{1}{n^2} m(E_n) \ldots (*)$$ (Note: m – normalised Lebesgue measure)

Define $f_n = \chi_{E_n} / \|\chi_{E_n}\|_{L^2}$ (Note: $\|\chi_{E_n}\|_{L^2} = m(E_n) > 0$)

Then $\|(M_\phi - \lambda I) f_n\|^2_{L^2} \leq \frac{1}{n}$ by $(*)$ above.

Hence there is a sequence $\{f_n\}$ of unit vectors such that $\|(M_\phi - \lambda I) f_n\|_{L^2} \to 0$ as $n \to \infty$

$\Rightarrow \lambda \in \Pi(M_\phi)$

i.e. $\text{ess ran}\, \phi \subset \Pi(M_\phi) \ldots$ $\underline{\underline{(A)}}$

~~Nowe~~ Now we establish that $\Pi(M_\phi) \subset \text{ess ran}\, \phi$

Let $\lambda \in \Pi(M_\phi)$ and assume $\lambda \notin \text{ess ran}\, \phi$

Then $\exists$ an $\epsilon > 0$ such that

$$m\{e^{i\theta} : |\phi(e^{i\theta}) - \lambda| < \epsilon\} = 0$$

so that $|\phi(e^{i\theta}) - \lambda| \geq \epsilon$ a.e.

This means $\dfrac{1}{|\phi - \lambda|} \leq \dfrac{1}{\epsilon}$ a.e.

so $\dfrac{1}{\phi - \lambda} \in L^\infty$

~~Let g = Then M_ϕ f~~

~~Then $(M_\phi - \lambda I) M$ f~~

Then $(M_\varphi - \lambda I) M_{\frac{1}{\varphi-\lambda}} = M_{(\varphi-\lambda)} \cdot M_{\frac{1}{(\varphi-\lambda)}}$

$= I$ (on L^2)

This means $\lambda \notin \sigma(M_\varphi)$

But then $\lambda \notin \Pi(M_\varphi)$ $(\because \Pi(M_\varphi) \subset \sigma(M_\varphi))$

This is a contradiction.

So our assumption that $\lambda \notin$ ess ran φ is wrong.

i.e. $\Pi(M_\varphi) \subset$ ess ran φ ... (B)

By (A) & (B) $\Pi(M_\varphi) =$ ess ran φ.

Definition Toeplitz Operator

For each $\phi \in L^\infty$, the Toeplitz operator with symbol ϕ is defined ~~by~~ from H^2 into H^2 by

$$T_\phi f = P(\phi f) \qquad \forall f \in H^2.$$

[Note: Here P is the analytic projection from L^2 onto H^2 i.e. if $h \in L^2$ and h is given by the Fourier series $\sum_{n=-\infty}^{\infty} \alpha_n e^{in\theta}$

then $$P(h) \sim \sum_{n=0}^{\infty} \alpha_n e^{in\theta}.$$]

<u>Theorem A</u> Let $\phi \in L^\infty$. Then the matrix of the Toeplitz operator T_ϕ w.r.t. the standard orthonormal basis $\{e^{in\theta}\}_{n=0}^{\infty}$ of H^2 is

$$\begin{bmatrix} \alpha_0 & \alpha_{-1} & \alpha_{-2} & \alpha_{-3} & \cdots \\ \alpha_1 & \alpha_0 & \alpha_{-1} & \alpha_{-2} & \cdots \\ \alpha_2 & \alpha_1 & \alpha_0 & \alpha_{-1} & \cdots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{bmatrix}$$

where $\phi \sim \sum_{n=-\infty}^{\infty} \alpha_n e^{in\theta}$

<u>Proof.</u> Let $[a_{m,n}]$ $\quad m = 0, 1, 2, \ldots$
$\quad n = 0, 1, 2, \ldots$

be the matrix of T_ϕ w.r.t. $\{e^{in\theta}\}_{n=0}^{\infty}$ on $H^2(\mathbb{T})$ i.e. H^2.

We have to show that for each non-negative integer k

$$a_{m,m+k} \text{ is } \alpha_{-k} \text{ for } m = 0, 1, 2$$

and

$$a_{m+k,m} \text{ is } \alpha_k \text{ for } m = 0, 1, 2 \ldots$$

Fix any non-negative k. Then for any fixed $m \geq 0$

~~and any $k \geq 0$~~

$$a_{m,m+k} = \langle T_\phi e^{i(m+k)\theta}, e^{im\theta} \rangle$$

$$= \langle P(\phi(e^{i\theta}) e^{i(m+k)\theta}, e^{im\theta} \rangle$$

$$= \left\langle P\left(\left(\sum_{n=-\infty}^{\infty} \alpha_n e^{in\theta}\right) e^{i(m+k)\theta}\right), e^{im\theta} \right\rangle$$

$$= \langle P(\ldots + \alpha_{-(m+k+1)} e^{-i\theta} + \alpha_{-(m+k)} + \alpha_{-(m+k)+1} e^{i\theta} + \ldots, e^{im\theta}$$

$$= \langle \alpha_{-m+1} + \alpha_{-(m+k)+1} e^{i\theta} + \dots , e^{im\theta} \rangle$$

$$= \alpha_{-k}$$

Hence for any m $\quad a_{m,m+k} = \alpha_{-k}$

Similarly it can be seen that for any m

$$a_{m+k,m} = \alpha_k$$

This proves the theorem.

<u>Definition</u> <u>Analytic Toeplitz Operator.</u>

If $\phi \in H^\infty$, then the Toeplitz operator T_ϕ is called an analytic Toeplitz operator.

Thus an analytic Toeplitz operator T_ϕ on H^2 is simply multiplication by ϕ on H^2.

Theorem B If T_φ is an analytic Toeplitz operator then the matrix of T_φ w.r.t. $\{e^{in\theta}\}_{n=0}^{\infty}$ is

$$\begin{bmatrix} \alpha_0 & 0 & 0 & \cdots & \\ \alpha_1 & \alpha_0 & 0 & 0 & \cdots \\ \alpha_2 & \alpha_1 & \alpha_0 & 0 & 0 \\ \vdots & & \ddots & \ddots & \ddots \end{bmatrix}$$

Proof. Just repeat the proof of Theorem A using the fact that $0 = \alpha_{-1} = \alpha_{-2} = \cdots$.

Theorem The Toeplitz operators on H^2 are the operators whose matrices w.r.t. the basis $\{e^{in\theta}\}_{n=0}^{\infty}$ of H^2 are Toeplitz matrices.

<u>Corollary</u> The operator $T \in B(H^2)$ is a Toeplitz operator if and only if $S^*TS = T$ where S is the unilateral shift.

<u>Proof</u> Let T be a Toeplitz operator. For any non-negative integers n and m

$$\langle S^*TS e^{in\theta}, e^{im\theta} \rangle$$

$$= \langle TS e^{in\theta}, S e^{im\theta} \rangle$$

$$= \langle T e^{i(n+1)\theta}, e^{i(m+1)\theta} \rangle$$

$$\cancel{= \langle T e^{in\theta}, e^{im\theta} \rangle \quad (\because \text{The matrix of } T \text{ is Toeplitz})}$$

$$\cancel{= \langle S^*TS e^{in\theta}, e^{im\theta} \rangle \quad (\because T = S^*TS)}$$

$$= \langle T e^{in\theta}, e^{im\theta} \rangle \quad (\because T \text{ has a Toeplitz matrix})$$

Hence $S^*TS = T$

<u>Conversely</u> Let $S^*TS = T$. We show T is a Toeplitz operator by showing T has a Toeplitz matrix.

For any $m, n \geq 0$ and for all natural numbers k

$$\langle Te^{i(n+k)\theta}, e^{i(m+k)\theta} \rangle$$

$$= \langle TSe^{i(n+k-1)\theta}, Se^{i(m+k-1)\theta} \rangle$$

$$= \langle S^*TSe^{i(n+k-1)\theta}, e^{i(m+k-1)\theta} \rangle$$

$$= \langle Te^{i(n+k-1)\theta}, e^{i(m+k-1)\theta} \rangle \quad (\because S^*TS = T)$$

$$= \langle TSe^{i(m+k-2)\theta}, Se^{i(m+k-2)\theta} \rangle$$

$$= \langle S^*TSe^{i(m+k-2)\theta}, e^{i(m+k-2)\theta} \rangle$$

$$= \langle Te^{i(m+k-2)\theta}, e^{i(m+k-2)\theta} \rangle \quad (\because S^*TS = T)$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$= \langle Te^{in\theta}, e^{im\theta} \rangle$$

Hence T has a Toeplitz matrix so that T is a Toeplitz operator.

<u>Theorem</u> The mapping $\varphi \to T_\varphi$ from L^∞ onto the space of Toeplitz operators on H^2 (where the space of Toeplitz operators is treated as as a subspace of the algebra $B(H^2)$) is 1-1, onto and closed under the adjoint operation i.e. if T_φ is Toeplitz then T_φ^* is Toeplitz and in fact $T_\varphi^* = T_{\bar{\varphi}}$.

<u>Proof</u>- Let $\varphi, \psi \in L^\infty$

Then for any scalars α, β and for all $m, n \geq 0$

$$\langle T_{\alpha\varphi+\beta\psi} e^{in\theta}, e^{im\theta} \rangle$$

$$= \langle P((\alpha\varphi+\beta\psi)e^{in\theta}), e^{im\theta} \rangle$$

$$= \alpha \langle P(\varphi e^{in\theta}), e^{im\theta} \rangle$$

$$+ \beta \langle P(\psi e^{in\theta}), e^{im\theta} \rangle$$

$$= \langle \alpha T_\varphi e^{in\theta}, e^{im\theta} \rangle + \langle \beta T_\psi e^{in\theta}, e^{im\theta} \rangle$$

$$= \langle (\alpha T_\varphi + \beta T_\psi) e^{in\theta}, e^{im\theta} \rangle$$

Hence $T_{\alpha\varphi+\beta\psi} = \alpha T_\varphi + \beta T_\psi$

so the map $\varphi \longrightarrow T_\varphi$ is linear.

Further

$$\| T_\varphi \| = \| P M_\varphi \| \qquad \text{(By definition of } T_\varphi)$$

$$\leq \| M_\varphi \| \qquad (\because \|P\| = 1)$$

$$= \| \varphi \|_\infty$$

Hence the map $\varphi \longrightarrow T_\varphi$ is bounded.

Finally, we establish that map preserves adjoints i.e. ~~if~~ for T_φ, $(T_\varphi)^*$ is also Toeplitz and in fact $T_\varphi^* = T_{\bar\varphi}$

Now, for any f, g in H^2

$$\langle T_\varphi^* f, g \rangle = \langle f, T_\varphi g \rangle$$

$$= \langle f, PM_\varphi g \rangle$$

$$= \langle f, PM_\varphi g \rangle + 0$$

$$= \langle f, PM_\varphi g \rangle + \langle f, (I-P)M_\varphi g \rangle$$

$(\because f \in H^2,\ (I-P)M_\varphi g \in H^{2\perp}$ so $\langle f, (I-P)M_\varphi g \rangle = 0)$

$$= \langle f, PM_\varphi g + (I-P)M_\varphi g \rangle$$

$$= \langle f, M_\varphi g \rangle \qquad (\because P + I - P = I)$$

$$= \langle f, \varphi g \rangle$$

$$= \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta})\, \overline{\varphi(e^{i\theta})\, g(e^{i\theta})}\, d\theta$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \bar{\varphi}(e^{i\theta})\, f(e^{i\theta})\, \overline{g(e^{i\theta})}\, d\theta$$

$$= \langle \bar{\varphi} f, g \rangle$$

$$= \langle (P + I - P)\bar{\varphi} f, g \rangle$$

$$= \langle P\bar{\varphi} f, g \rangle + \langle (I-P)\bar{\varphi} f, g \rangle$$

$$= \langle P(\bar{\varphi} f), g \rangle$$

$\left(\because (I-P)\bar{\varphi} f \in {H^2}^{\perp} \right.$
and $g \in H^2$
so $\left. \langle (I-P)\bar{\varphi} f, g \rangle = 0 \right)$

$$= \langle T_{\bar{\varphi}} f, g \rangle$$

Hence $T_{\varphi}^{*} = T_{\bar{\varphi}}$

TOEPLITZ OPERATORS (continued)

Definition Let H be a Hilbert space and let f, g be two elements of H.

The operator $f \otimes g : H \longrightarrow H$ is defined as

$$(f \otimes g) h = \langle h, g \rangle f$$

Clearly If $f \neq 0 \neq g$ then the range of the operator $f \otimes g$ is the one dimensional subspace $\langle f \rangle$ of H.

Also $\quad f \otimes g = 0 \iff f = 0$ or $g = 0$.

Theorem T1. If $A, B \in B(H)$ and $f, g \in H$ then $\quad A(f \otimes g) B = (Af) \otimes (B^* g)$

Proof For any h in H

$$\begin{aligned}(A(f \otimes g) B)(h) &= (A(f \otimes g)) Bh \\ &= A(f \otimes g)(Bh) = A(\langle Bh, g \rangle f)\end{aligned}$$

$$= \langle Bh, g \rangle Af$$

$$= \langle h, B^*g \rangle Af$$

$$= (Af \otimes B^*g)\, h \qquad \text{(By the definition of } \otimes\text{)}$$

Hence $\quad A(f \otimes g)B = Af \otimes B^*g$

<u>Theorem T2.</u> If $\psi, \phi \in L^\infty$ and T_ψ, T_ϕ are the corresponding Toeplitz operators on H^2 and S is the unilateral shift on H^2 then

$$S^* T_\psi T_\phi S - T_\psi T_\phi = P(e^{-i\theta}\psi) \otimes P(e^{-i\theta}\bar{\phi})$$

where P is the orthogonal projection from L^2 onto H^2

<u>Proof.</u> We first observe that

$$(SS^* + 1 \otimes 1)(e^{in\theta}) \qquad (n = 0, 1, 2, \ldots)$$

$$= \begin{cases} 0 + 1 & \text{if } n = 0 \\ e^{in\theta} & \text{if } n = 1, 2, \ldots \end{cases}$$

$(\because\ SS^*1 = 0,\ SS^*e^{in\theta} = e^{in\theta},\ n = 1, 2, 3, \ldots$ and $(1 \otimes 1)(1) = 1$ $\quad n = 1, 2, 3$

i.e. $SS^* + 1 \otimes 1 = I$ on H^2.

Hence

$$
\begin{aligned}
S^* T_\psi T_\phi S &= S^* T_\psi I T_\phi S \\
&= S^* T_\psi (SS^* + 1 \otimes 1) T_\phi S \qquad \text{(By Thm T2.)} \\
&= S^* T_\psi S S^* T_\phi S + S^* T_\psi (1 \otimes 1) T_\phi S \\
&= T_\psi T_\phi + S^* T_\psi (1 \otimes 1) T_\phi S
\end{aligned}
$$

($\because$ T is a Toeplitz operator $\Leftrightarrow S^* T S = T$ ~~and by~~ Th~~eorem T1~~)

$$
\begin{aligned}
&= T_\psi T_\phi + (S^* T_\psi 1) \otimes (S^* T_{\bar\phi} 1) \qquad \text{(By Thm. T} \\
&= T_\psi T_\phi + (T_{\bar\psi} S)^* 1 \otimes (T_\phi S)^* 1
\end{aligned}
$$

($\because$ $T_\psi^* = T_{\bar\psi}$ and $(AB)^* = B^* A^*$)

$$
= T_\psi T_\phi + (T_{e^{i\theta}\bar\psi})^* 1 \otimes (T_{e^{i\theta}\phi})^* 1
$$

($\because$ $T_{\bar\psi} S h = T_{\bar\psi} e^{i\theta} h = P(\bar\psi e^{i\theta} h) = T$[illegible]$\, h = T$[illegible]$\, h$)

$$= T_\psi T_\varphi + \left(T_{e^{-i\theta}\psi} 1 \otimes T_{e^{-i\theta}\bar{\varphi}} 1 \right)$$

$$\left(\because T_h^* = T_{\bar{h}} \right)$$

$$= T_\psi T_\varphi + \left(P(e^{-i\theta}\psi) \otimes P(e^{-i\theta}\bar{\varphi}) \right)$$

$(\because$ By definition $T_h 1 = P(h))$

$$\therefore \quad S^* T_\psi T_\varphi S - T_\psi T_\varphi = P(e^{-i\theta}\psi) \otimes P(e^{-i\theta}\bar{\psi}$$

Hence proved.

Definition Co-analytic Toeplitz Operator

We know that for $\varphi \in L^\infty$ we say T_φ is an analytic Toeplitz operator if $\varphi \in H^\infty$

We say T_φ (for $\varphi \in L^\infty$) is co-analytic if T_φ^* is an analytic Toeplitz operator

But $T_\varphi^* = T_{\bar{\varphi}}$

So T_φ is co-analytic if $\bar{\varphi} \in H^\infty$

Products of Toeplitz Operators

Theorem T3 Let ψ and φ belong to L^∞.

Then $T_\psi T_\varphi$ is a Toeplitz operator

$\Longleftrightarrow$

Either T_ψ is coanalytic or T_φ is analytic

Note: If $T_\psi T_\varphi$ is a Toeplitz operator then

$$T_\psi T_\varphi = T_{\psi\varphi}$$

Proof. Let T_φ be analytic.

Then $T_\psi T_\varphi f = T_\psi P(\varphi f) \qquad \forall f \in H^2$

$= T_\psi(\varphi f) \qquad (\because \varphi f \in H^2$ as $\varphi \in H^\infty)$

$$= P(\psi\varphi f)$$

$$= T_{\psi\varphi} f$$

$\therefore$ $T_\psi T_\varphi = T_{\psi\varphi}$ and so $T_\psi T_\varphi$ is Toeplitz.

When ~~T_ψ is~~ T_ψ is co-analytic

we have $\bar{\psi} \in H^\infty$

Now $(T_\psi T_\varphi)^* = T_\varphi^* T_\psi^*$

$$= T_{\bar\varphi} T_{\bar\psi}$$

$$= T_{\bar\varphi\bar\psi} \quad (\because T_{\bar\psi} \text{ is analytic and by the first part of the proof above})$$

$\therefore$ $T_\psi T_\varphi = (T_{\bar\varphi\bar\psi})^*$

$$= T_{\varphi\psi}$$

$$= T_{\psi\varphi}$$

so $T_\psi T_\varphi$ is a Toeplitz operator.

Conversely assume $T_\psi T_\varphi$ is a Toeplitz operator

By Theorem T2 for the unilateral shift S

(*) $$S^* T_\psi T_\varphi S - T_\psi T_\varphi = P(\bar{e}^{i\theta}\psi) \otimes P(\bar{e}^{i\theta}\bar{\varphi})$$

But since $T_\psi T_\varphi$ is a Toeplitz operator

and since any operator T is Toeplitz

if and only if $S^* T S = T$ (*) becomes

$$T_\psi T_\varphi - T_\psi T_\varphi = P(\bar{e}^{i\theta}\psi) \otimes P(\bar{e}^{i\theta}\bar{\varphi})$$

i.e. $P(\bar{e}^{i\theta}\psi) \otimes P(\bar{e}^{i\theta}\bar{\varphi}) = 0$

This means either $P(\bar{e}^{i\theta}\psi) = 0$

OR

$$P(\bar{e}^{i\theta}\bar{\varphi}) = 0$$

If $P(\bar{e}^{i\theta}\psi) = 0$ then if $\psi = \sum_{-\infty}^{\infty} \alpha_n e^{in\theta}$

$$\bar{e}^{i\theta}\psi = \cdots + \alpha_{-1}\bar{e}^{2i\theta} + \alpha_0 \bar{e}^{i\theta} + \alpha_1 + \alpha_2 e^{i\theta} + \cdots$$

so $P(e^{-i\theta}\psi) = \alpha_1 + \alpha_2 e^{i\theta} + \alpha_3 e^{2i\theta} + \dots$

But since $P(e^{-i\theta}\psi) = 0$ we get

$$0 = \alpha_1 = \alpha_2 = \alpha_3 = \dots$$

so $\psi = \alpha_0 + \alpha_{-1} e^{-i\theta} + \alpha_{-2} e^{-2i\theta} + \dots$

so $\overline{\psi} = \bar{\alpha}_0 + \bar{\alpha}_{-1} e^{i\theta} + \bar{\alpha}_{-2} e^{2i\theta} + \dots$

i.e. $T_{\bar{\psi}}$ is analytic or T_ψ is co-analytic

If $P(e^{-i\theta}\bar{\varphi}) = 0$ then by the same reasoning as above for ψ we get

$$\overline{\varphi} = \beta_0 + \beta_{-1} e^{-i\theta} + \beta_{-2} e^{-2i\theta} + \dots$$

so $\varphi = \bar{\beta}_0 + \bar{\beta}_{-1} e^{i\theta} + \bar{\beta}_{-2} e^{2i\theta} + \dots$

so $\varphi \in H^\infty$ and T_φ is analytic.

<u>Corollary T_4</u> The product of two Toeplitz operators is zero if and only if one of the factors is zero.

<u>Proof.</u> Let T_ψ, T_φ be two Toeplitz operators such that $T_\psi T_\varphi = 0$

Now the operator that takes every element of H^2 to 0 i.e. the zero operator is trivially a Toeplitz operator so $T_\psi T_\varphi = 0$ implies that $T_\psi T_\varphi$ is a Toeplitz operator.

Thus either T_ψ is co-analytic or T_φ is analytic and also the product $T_\psi T_\varphi$ being Toeplitz, must by Thm T_3 be $T_{\psi\varphi}$

Hence $T_{\psi\varphi} = 0$ so $\psi\varphi = 0$

But if $T_\varphi \not\equiv 0$ then $\varphi \not\equiv 0$ and so since $\varphi \in H^\infty \subset H^2$, φ cannot vanish on any set of +ve Lebesgue measure

So from $\psi\varphi = 0$ we conclude $\psi \equiv 0$ which means $T_\psi = 0$

If $T_{\bar{\psi}}$ is analytic but not zero then $\bar{\psi} \not\equiv 0$ and as $\bar{\psi} \in H^\infty \subset H^2$ $\bar{\psi}$ cannot vanish on any set of +ve Lebesgue measure and so from

$$\psi\varphi = 0 \quad \text{we get} \quad \varphi \equiv 0$$

so $T_\varphi = 0$.

Theorem T_5

Let $\varphi, \psi \in L^\infty$. Then $T_\varphi T_\psi = T_\psi T_\varphi$ if and only if one of the following holds:

(i) Both T_φ, T_ψ are analytic Toeplitz operato

(ii) Both T_φ, T_ψ are co-analytic Toeplitz operators

(iii) $\exists$ complex numbers a, b (at least one of them is not zer such that $a\varphi + b\psi$ is a constant.

Corollarly T_6

If two Toeplitz operators commute with each other and neither is a linear combinatio of the identity andofthe other operator Then their product is a Toeplitz operator.

Proof. By Theorem T_5, since the two Toeplitz operators commute and since by the conditions of Cor. T_6, the condition (iii) of Theorem T_5 is ruled out

so either both are analytic
or both are co-analytic.

In either case their product shall be
a Toeplitz operator.

If φ, ψ are in H^∞ i.e. T_φ, T_ψ are analytic

then $T_\varphi T_\psi f = T_\varphi P(\psi f) = T_\varphi(\psi f)$

$(\because \psi f \in H^2)$

$= P(\varphi \psi f)$

$= \varphi\psi f \quad (\because \varphi\psi f \in H^2)$

$= T_{\varphi\psi} f.$

so $T_\varphi T_\psi = T_{\varphi\psi}$ is a Toeplitz operator

Similarly if they are both co-analytic Toeplitz

$T_{\bar\varphi}, T_{\bar\psi}$ then $T_{\bar\varphi} T_{\bar\psi} = (T_\varphi)^*(T_\psi)^*$

$= (T_\psi T_\varphi)^* = (T_{\psi\varphi})^*$

[illegible] Toeplitz $\longleftarrow$ $= T_{\bar\psi\bar\varphi}$

TOEPLITZ OPERATORS (continued)

Theorem Let T_φ be a Toeplitz operator on H^2 for some $\varphi \in L^\infty$. Then $T_\varphi^* = T_{\bar{\varphi}}$

(Proved earlier in the notes)

Corollary A Toeplitz operator T_φ is self-adjoint if and only its symbol φ is real-valued almost everywhere.

Proof. T_φ is self adjoint

$$\Longleftrightarrow \quad T_\varphi^* = T_\varphi$$

$$\Longleftrightarrow \quad T_{\bar{\varphi}} = T_\varphi \qquad (\because\ T_\varphi^* = T_{\bar{\varphi}})$$

$$\Longleftrightarrow \quad \bar{\varphi} = \varphi \qquad (\because\ T_{\bar{\varphi}} = T_\varphi \Leftrightarrow T_{\bar{\varphi}} - T_\varphi = 0 \Leftrightarrow T_{\bar{\varphi}-\varphi} = 0 \Leftrightarrow \bar{\varphi} - \varphi = 0)$$

$$\Longleftrightarrow \quad \varphi \text{ is real-valued a.e.}$$

Hence proved.

Corollary (to Theorem T)

The Toeplitz operator T_φ is normal if and only if

$\varphi = \alpha\psi + \beta$ where $\psi \in L^\infty$ and is real valued and α, β are complex numbers.

Proof. If $\varphi = \alpha\psi + \beta$ as above then

since $T_\varphi^* = T_{\bar\varphi} = T_{\bar\alpha\psi + \bar\beta} = \bar\alpha T_\psi + \bar\beta I$

and $T_\varphi = T_{\alpha\psi+\beta} = \alpha T_\psi + \beta I$

We find that

$$T_\varphi^* T_\varphi = (\bar\alpha T_\psi + \bar\beta I)(\alpha T_\psi + \beta I)$$

$$= (\alpha T_\psi + \beta I)(\bar\alpha T_\psi + \beta I)$$

$$= T_\varphi T_\varphi^*$$ so T_φ is normal

Conversely

Assume T_φ is normal.

Then $T_\varphi T_\varphi^* = T_\varphi^* T_\varphi$

i.e. $T_\varphi T_{\bar\varphi} = T_{\bar\varphi} T_\varphi$

By Theorem T__, at least one of the following 3 cases holds

(i) Both T_φ and $T_{\bar\varphi}$ are analytic

(ii) Both T_φ and $T_{\bar\varphi}$ are co-analytic

(iii) There exist complex numbers α, β such that $\alpha\varphi + \beta\bar\varphi$ is a constant.

If (i) holds then φ and $\bar\varphi$ are in $H^\infty \subset H^2$

This means φ is a constant

If (ii) holds then φ and $\bar\varphi$ are $\overline{H^\infty}$ (complex conjugat

and again This means φ is a constant

If (iii) holds then there are constants α, β such that $\alpha\varphi + \beta\bar\varphi = \delta$ for some constant δ.

and so $\bar\alpha\bar\varphi + \bar\beta\varphi = \bar\delta$ (By taking conjugates of the above)

Adding the above two equations we have

$$(\alpha + \bar\beta)\varphi + (\bar\alpha + \beta)\bar\varphi = \delta + \bar\delta$$

i.e. $2\,\mathrm{Re}\,(\alpha+\bar\beta)\varphi = \delta + \bar\delta \Rightarrow \mathrm{Re}\,(\alpha+\bar\beta)\varphi = \frac{\delta+\bar\delta}{2}$

Then ~~Let~~ $(\alpha+\bar\beta)\varphi = \frac{\delta+\bar\delta}{2} + i\psi$, where ψ is the imaginary part of $(\alpha+\bar\beta)\varphi$

obviously ψ is real valued

If $\alpha + \bar{\beta} \neq 0$ then the result stands proved.

If $\alpha + \bar{\beta} = 0$

i.e if $\alpha = -\bar{\beta} \Rightarrow \beta = -\bar{\alpha}$

then since $\alpha\varphi + \beta\bar{\varphi} = \delta$

we have $\alpha\varphi - \bar{\alpha}\bar{\varphi} = \delta$

which means $\operatorname{Im}\varphi = \dfrac{\delta}{2i} = \text{constant}$

so $\varphi = \operatorname{Re}\varphi + i\,\dfrac{\delta}{2i}$

$= \operatorname{Re}\varphi + \dfrac{\delta}{2}$

Putting $\psi = \operatorname{Re}\varphi$ the result is proved.

<u>Theorem</u> The only compact Toeplitz operator is 0.

<u>Proof</u>. Recall that if an operator T on a Hilbert space H is compact then if $\{x_n\}$ is a weakly convergent sequence in H, then $\{Tx_n\}$ is a norm convergent sequence.

Now, for each non-negative integer k, the sequence $\{e^{i(k+n)\theta}\}$ converges weakly to zero as $n \to \infty$.

$$\left(\because \langle e^{i(k+n)\theta}, f \rangle \to 0 \text{ as } n \to \infty \ \forall f \text{ in } H^2 \right)$$

Hence if T_φ is compact, $\{T_\varphi e^{i(k+n)\theta}\}$ is norm convergent and so for each fixed k, l the sequence $\{\langle T_\varphi e^{i(k+n)\theta}, e^{i(l+n)\theta} \rangle\}$ converges to zero as $n \to \infty$.

But for each fixed n, because T_φ is a Toeplitz operator,

$$(*) \qquad \langle T_\varphi e^{i(k+n)\theta}, e^{i(l+n)\theta} \rangle = \alpha_{k-l}$$

where $\varphi \sim \sum_{n=-\infty}^{\infty} \alpha_n e^{in\theta}$

Now the L.H.S. of $(*)$ converges to zero as $n \to \infty$

But the R.H.S remains constant as α_{k-l}

So for each fixed +ve integers k, l, $\alpha_{k-l} = 0$

But this means $\alpha_m = 0 \quad \forall\, m \in \mathbb{Z}$

($\because$ if k, l vary over $1, 2, 3, \ldots$ then $k-l$ varies over $\mathbb{Z}$).

This means $\phi \equiv 0$

Hence proved.